Midterm Examination

CS 205A: Mathematical Methods for Robotics, Vision, and Graphics (Spring 2017),
Stanford University

- The exam runs for 75 minutes.
- The exam contains five problems. **You must complete all problems.**
- The exam is closed book/notes. You may use one double-sided $8^1/_2'' \times 11''$ sheet of notes. No calculators may be used.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem and indicate that you have done so.
- Do not spend too much time on any problem. Read them all before beginning.
- Show your work, as partial credit will be awarded.

Problem	1	2	3	4	5	Total
Score (/10 each)						

The Stanford Honor Code

1. The Honor Code is an undertaking of the students, individually and collectively:

    (a) that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

    (b) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

2. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

3. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

Signature

Name

SUID

Problem 1 (True or False).

$$\text{True/False grading scheme: } \begin{cases} +1 & \text{if correct,} \\ 0 & \text{if unanswered,} \\ -1 & \text{if incorrect.} \end{cases}$$

(1) The eigenvalue decomposition of a *symmetric* matrix is also its singular value decomposition.

(2) The eigenvalues of a square orthogonal matrix are all equal to 1.

(3) The singular values of a square orthogonal matrix are all equal to 1.

(4) Given a square matrix, M, the matrices $MM^\top$ and $M^\top M$ have the same eigenvalues.

(5) The condition number of a very well-conditioned matrix can be less than 1.

(6) The inverse of an upper triangular matrix is upper triangular.

(7) The eigenvalues of Hermitian matrices are real.

(8) The Cholesky factorization always exists for a symmetric matrix.

(9) If a QR factorization exists for a symmetric matrix, that matrix is invertible.

(10) $Volume(\|\vec{x}\|_1 \leq 1) \leq Volume(\|\vec{x}\|_\infty \leq 1)$

Problem 2 (Short Answers).

(a) **Sherman-Morrison-Woodbury and LU**: The Sherman-Morrison-Woodbury (SMW) formula says the inverse of a matrix A after a rank-one modification is given by

$$(A + \vec{u}\vec{v}^T)^{-1} = A^{-1} - \frac{1}{1 + \vec{v}^T A^{-1} \vec{u}} A^{-1} \vec{u} \vec{v}^T A^{-1}.$$

Suppose that you want to solve $(A + \vec{u}\vec{v}^T)\vec{x} = \vec{b}$ for some new $\vec{u}$,$\vec{v}$ and $\vec{b}$, but you already have an LU factorization of $A = LU$. Using the SMW formula, write a brief algorithm to compute $\vec{x}$ involving the fewest LU solves. (Hint: Define intermediate variables.) [5 points]

(b) **"H" is for Householder!** Consider the 3x3 matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. Using your knowledge of Householder reflections, determine a Householder reflection vector $\vec{v}$ such that the orthogonal reflection matrix, $H = I - \frac{2}{\|\vec{v}\|_2^2}\vec{v}\vec{v}^\top$, will produce the following sparsity pattern: $HA = \begin{bmatrix} \times & 0 & \times \\ \times & \times & \times \\ \times & 0 & \times \end{bmatrix}$. (Note: You do not need to compute H or HA.) [5 points]

Problem 3 (Matrix Factorizations).

(a) For a wide matrix $A \in \mathbb{R}^{m \times n}$ with $m < n$, the under-determined least-squares problem $A\vec{x} = \vec{b}$ is usually not solved by forming $A = QR$. Why? [2 points]

(b) Instead, you can form the $A = \bar{L}\bar{Q}$ factorization, where $\bar{L}$ is lower triangular, and $\bar{Q}$ has orthogonal rows. Construct this factorization by first computing the QR factorization of $A^\top$, and transposing the factorization–what are $\bar{L}$ and $\bar{Q}$? [3 points]

(c) Solve the under-determined $A\vec{x} = \vec{b}$ problem using the $\bar{L}\bar{Q}$ factorization. [2 points]

(d) Given a Cholesky factorization of a symmetric matrix, $A = LL^T$, show how to construct the LDLT factorization, $A = \tilde{L}D\tilde{L}^T$, where $\tilde{L}$ is unit lower triangular (ones on the diagonal), and D is a diagonal matrix. [3 points]

Problem 4 (Matrix two-norm and SVD).

For a general matrix $A \in \mathbb{R}^{m \times n}$, use its singular value decomposition, $A = U\Sigma V^\top$, to prove that the induced-2 norm $\|A\|_2$ is equal to A's maximum singular value, σ_1. [10 points]

Problem 5 (Condition Numbers).

A Guessing Game: The great Vizzini[1] has challenged you to a life-or-death battle of wits on condition numbers of symmetric matrices. Inconceivable! Are you ready? Good. You must guess an integer, x. But since he doesn't have all day, he's willing to tell you three other numbers: a, b and c, where c is the two-norm condition number of a diagonal 3x3 matrix, $A = \text{diag}(a, b, x)$.

"If you think this is too easy, you're wrong! **You only have four tries!** Ha ha ha ha ha! <nearly falls on ground> Go!"

(a) Stalling for time, you first derive an expression for the two-norm condition number,

$$c(x) = \text{cond}\,(A(x))$$

on a napkin–actually, just write it on the exam, please. [5 points]

[1] A super-dated villain from Prof. James' childhood.

(b) If the numbers he gives you are “2, 3 and 6,” is it conceivable that you will be able to determine the answer in just *four* guesses? Why? (Hint: Plot $c(x)$.) [3 points]

(c) What about if he says “2, 4 and ... 2. Ha ha ha!” Why? [2 points]

(d) *Bonus:* How many tries would you need in (c) to be sure? Remember this is life or death! [2 points. You can guess if you want ;)]