Midterm Examination

CS 205A: Mathematical Methods for Robotics, Vision, and Graphics (Fall 2013),
Stanford University

- The exam runs for 75 minutes.
- The exam contains seven problems. You must complete the first problem and five of problems 2-7. **CIRCLE THE PROBLEMS YOU WANT GRADED ON THE CHART BELOW; OTHERWISE WE WILL GRADE THE FIRST FIVE QUESTIONS ON WHICH YOU HAVE PROVIDED ANY WRITTEN ANSWER.**
- The exam is closed book/notes. You may use one double-sided $8^1/_2'' \times 11''$ sheet of notes.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem and indicate that you have done so.
- Do not spend too much time on any problem. Read them all before beginning.
- Show your work, as partial credit will be awarded.

Circle the five additional problems you want graded.

Problem	1	2	3	4	5	6	7	Total
Score (out of 10)								

The Stanford Honor Code

1. The Honor Code is an undertaking of the students, individually and collectively:

   (a) that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

   (b) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

2. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

3. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

Signature

Name

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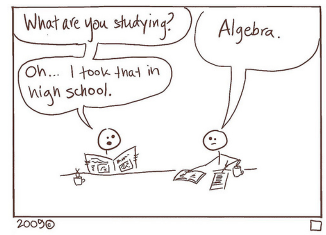
YOU MUST COMPLETE THIS PROBLEM.

Problem 1 (Short answer).

(a) Derive a linear system of equations satisfied by minima of $f(\vec{x}) \equiv \|A\vec{x} - \vec{b}\|_2$ with respect to $\vec{x}$, for $A \in \mathbb{R}^{m\times n}$. [2 points]



(b) When might it be preferable to use a fixed-point representation of real numbers over floating-point? [2 points]

(c) Draw a picture of two vectors in $\mathbb{R}^2$ illustrating why Gram-Schmidt orthogonalization can be numerically unstable; explain what can go wrong. [2 points]

(d) Provide a variational problem solved by computing one or more eigenvalues of $A^\top A$ for $A \in \mathbb{R}^{m\times n}$. [2 points]

(e) Is Newton's method guaranteed to have quadratic convergence? Why? [2 points]

CIRCLE WHICH FIVE OF PROBLEMS 2-7 YOU WANT GRADED ON PAGE 1.

Problem 2 (Screened Poisson smoothing). Suppose we sample a function $f(x)$ at n positions $x_1, x_2, \ldots, x_n$, yielding a point $\vec{y} \equiv (f(x_1), f(x_2), \ldots, f(x_n)) \in \mathbb{R}^n$. Our measurements might be noisy, however, so a common task in graphics and statistics is to smooth these values to obtain a new vector $\vec{z} \in \mathbb{R}^n$.

(a) Provide least-squares energy terms measuring the following:

(i) The similarity of $\vec{y}$ and $\vec{z}$. [1 point]

(ii) The smoothness of $\vec{z}$. [3 points]
Hint: We expect $f(x_{i+1}) - f(x_i)$ to be small for smooth f.

(b) Propose an optimization problem for smoothing $\vec{y}$ using the terms above to obtain $\vec{z}$, and argue that it can be solved using linear techniques. [3 points]

(c) Suppose n is very large. What properties of the matrix in (b) might be relevant in choosing an effective algorithm to solve the linear system? [3 points]

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Problem 3 (Tracking infections). Suppose $\vec{x}_0 \in \mathbb{R}^n$ contains sizes of different populations carrying a particular infection in year 0; for example, when tracking malaria we might take x_{01} to be the number of humans with malaria and x_{02} to be the number of mosquitoes carrying the disease. By writing relationships like "The average mosquito infects two humans" we can write a matrix M such that $\vec{x}_1 \equiv M\vec{x}_0$ predicts populations in year 1, $\vec{x}_2 \equiv M^2\vec{x}_0$ predicts populations in year 2, and so on.

(a) Recall that the spectral radius $\rho(M)$ is given by $\max_i |\lambda_i|$, where the eigenvalues of M are $\lambda_1, \ldots, \lambda_k$. Epidemiologists call this number the "reproduction number" $\mathcal{R}_0$ of M. Explain the difference between the cases $\mathcal{R}_0 < 1$ and $\mathcal{R}_0 > 1$ in terms of the spread of disease. Which case is more dangerous? [5 points]

(b) Suppose we only care about proportions. For instance, we might use $M \in \mathbb{R}^{50\times 50}$ to model transmission of diseases between residents in each of the 50 states of the USA, and we only care about the fraction of the total people with a disease who live in each state. If $\vec{y}_0$ holds these proportions in year 0, give an iterative scheme to predict proportions in future years. Characterize behavior as time goes to infinity. [5 points]

NOTE: Those students concerned about computer graphics applications of CS 205A should know that the reproduction number $\mathcal{R}_0$ is referenced in the 2011 thriller *Contagion*.

CIRCLE WHICH FIVE OF PROBLEMS 2-7 YOU WANT GRADED ON PAGE 1.

Problem 4 (LU factorization).

(a) Suppose $A \in \mathbb{R}^{n\times n}$ admits a Cholesky factorization $A = LL^\top$.

(i) Show that A must be positive semidefinite. [3 points]

(ii) Use this observation to suggest an algorithm for checking if a matrix is positive semidefinite. [2 points]

(b) Show that any invertible matrix $A \in \mathbb{R}^{n\times n}$ with $a_{11} = 0$ cannot have a factorization $A = LU$ for lower triangular L and upper triangular U. [5 points]

Hint: Suppose such a factorization exists, and write a_{11} in terms of entries of L and U.

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Problem 5 (Eigenvalue problems).

(a) Suppose $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$ such that $\vec{u}^\top \vec{v} = 1$, and define $A \equiv \vec{u}\vec{v}^\top$.

(i) What are the eigenvalues of A? [2 points]

(ii) How many iterations does power iteration take to converge to the dominant eigenvalue of A? [3 points]

(b) Suppose $B \in \mathbb{R}^{n\times n}$ is diagonalizable with eigenvalues λ_i satisfying $0 < \lambda_1 = \lambda_2 < \lambda_3 < \cdots < \lambda_n$. Let $\vec{v}_i$ be the eigenvector corresponding to λ_i. Show that the inverse power method applied to B converges to a linear combination of $\vec{v}_1$ and $\vec{v}_2$. [5 points]

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Problem 6 (Singular Value Decomposition).

(a) Provide the SVD and condition number with respect to $\| \cdot \|_2$ of the following matrices. [4 points]

(i) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ \sqrt{3} & 0 & 0 \end{pmatrix}$

(ii) $\begin{pmatrix} -5 \\ 3 \end{pmatrix}$

(b) Show that $\|A\|_2 = \|\Sigma\|_2$, where $A = U\Sigma V^T$ is the singular value decomposition of A. [3 points]

(c) Show that the null space of a matrix $A \in \mathbb{R}^{n\times n}$ is spanned by columns of V corresponding to zero singular values, where $A = U\Sigma V^\top$ is the singular value decomposition of A. [3 points]

CIRCLE WHICH FIVE OF PROBLEMS 2-7 YOU WANT GRADED ON PAGE 1.

Problem 7 (QR).

(a) Take $A \in \mathbb{R}^{m\times n}$ and suppose we apply the Cholesky factorization to obtain $A^\top A = LL^\top$. Define $Q \equiv A(L^\top)^{-1}$. Show that Q is orthogonal. [3 points]

(b) Based on the previous part, suggest a relationship between the Cholesky factorization of $A^\top A$ and QR factorization of A. [2 points]

(c) Suppose $A \in \mathbb{R}^{m\times n}$ is rank m with $m < n$. Suppose we factor

$$A^\top = Q \begin{pmatrix} R_1 \\ 0 \end{pmatrix}.$$

Provide a solution $\vec{x}$ to the underdetermined system $A\vec{x} = \vec{b}$ in terms of Q and R_1. [5 points]

Hint: Try the square case $A \in \mathbb{R}^{n\times n}$ first, and use the result to guess a form for $\vec{x}$. Be careful that you multiply matrices of proper size.