Homework 1: Numerics and Linear Algebra (LU)

CS 205A: Mathematical Methods for Robotics, Vision, and Graphics (Winter 2018)
Stanford University
Due Thursday, Jan 25, before 11:59 PM (via gradescope)

> This homework is one of the more difficult assignments in CS 205A. Please make ample use of office hours, Piazza, and other resources, and get started early.
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> Note: As mentioned in lecture, although it is not worth course credit you may wish as an exercise on your own to implement algorithms we discuss in class. We are happy to help you debug and experiment with your implementations in office hours and on Piazza. From last weeks lectures, the obvious choices for implementation are Gaussian elimination (with or without pivoting) and LU factorization.

Textbook problems: : 2.5 (20 points), 3.12(a-e) (35 points)

Julia Programming Assignment (25 points): In this question, you will explore how the condition number of a matrix can have practical influence on which algorithms (e.g. LU, Cholesky) you can use to solve linear systems of the form: $A\overrightarrow{x} = \overrightarrow{b}$ where $A \in \mathrm{R}^{nxn}$. We will study the Vandermonde matrix, as well as a matrix constructed from a finite Fourier Series basis.
In this question, you will interpolate samples of the analytical function

$$f(x) = \frac{1}{1+x^2}$$

for $x \in [0,1]$. Your monomial interpolants (with $N = n+1$ terms) are given by

$$g_V(x) = \sum_{j=0}^{n} c_j x^j$$

and

$$g_F(x) = \sum_{j=1}^{N/2} c_j sin(j\pi x) + \sum_{j=N/2+1}^{N} c_j cos((j-N/2)\pi x)$$

Use $M = m+1$ uniformly sampled positions

$$x_i = ih,\ i = 0...m,$$

with spacing $h = 1/m$, to generate M samples of the test function f,

$$f_i = f(x_i), i = 0...m.$$

To estimate the polynomial coefficients $\overrightarrow{c}$, you will assemble and solve the linear systems

$$V\overrightarrow{c} = \overrightarrow{f}$$

and

$$F\overrightarrow{c} = \overrightarrow{f}$$

where V is the M-by-N Vandermonde matrix with entries

$$V_{ij} = (x_i)^j$$

and F is the finite Fourier Series basis matrix with entries

$$F_{i,j-1} = \begin{cases} sin(j\pi x_i), & \text{if } 1 \leq j \leq N/2 \\ cos((j - N/2)\pi x_i), & N/2 + 1 \leq j \leq N \end{cases}$$

For simplicity, assume M = N for the entire problem.

1. (10 points) Use an LU solve to estimate the monomial coefficients $\overrightarrow{c}$. Report the residual L2 norm for both linear systems when $N = 8$ and $N = 16$.

2. (4 points) Using the cond() function in Julia, plot N vs. $cond(V)$ and N vs. $cond(F)$ for $N = 4, 6, 8, ...32$. Write a couple of sentences explaining the reasons for the trends in these two plots. Hints: Use a logarithmic scale in y axis for better clarity. Also, try wrapping the creation of V and F into functions that you can call repeatedly to generate the required output data. These functions will be helpful for the next part.

3. (6 points) A necessary condition for being able to use Cholesky factorization is that the matrix must be positive definite. Construct $A_V = V^T V$ and $A_F = F^T F$ for $N = 4, 6, ...32$ (a total of 30 matrices). Mathematically, when would these matrices be positive definite? Explain.
Using the isposdef() function, check to which matrices Julia reports as positive definite. Create a table of values that includes the following columns: $N, isposdef(A_v), isposdef(A_F), cond(V), cond(F)$.
What is the largest value of N where A_V is positive definite, and what is the condition number of that V? What is the largest value of N where A_F is positive definite, and what is the condition number of that F? Are these condition numbers connected in some way? If so, how?

4. (5 points) For $N = 8$, use Cholesky factorization to solve the two linear systems. Report the residual L2 norm for each solution. Compare the residuals to Part 1: how does Cholesky compare to LU?

To simplify submission to GradeScope with your other written homework, export a PDF of a clearly documented Julia Notebook that shows your work.