Nonlinear Systems

CS 205A:
Mathematical Methods for Robotics, Vision, and Graphics

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Part III: Nonlinear Problems

Not all numerical problems can be solved with \ in Matlab.

Question

Have we already seen a nonlinear problem?

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$$\text{minimize } \|A\vec{x}\|_2$$
$$\text{such that } \|\vec{x}\|_2 = 1 \longleftarrow \text{nonlinear!}$$

Root-Finding Problem

$$\text{Given: } f : \mathbb{R}^n \to \mathbb{R}^m$$

$$\text{Find: } \vec{x}^* \text{ with } f(\vec{x}^*) = \vec{0}$$

Root-Finding Applications

- Collision detection (graphics, astronomy)
- Graphics rendering (ray intersection)
- Robotics (kinematics)
- Optimization (line search)

Issue: Regularizing Assumptions

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$$g(x) = \begin{cases} -1 & \text{when } x \in \mathbb{Q} \\ 1 & \text{when } x \notin \mathbb{Q} \end{cases}$$

Typical Regularizing Assumptions

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C^k

k derivatives exist and are continuous

Today

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

Property of Continuous Functions

Intermediate Value Theorem

Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $f(a) < u < f(b)$ or $f(b) < u < f(a)$. Then, there exists $z \in (a, b)$ such that $f(z) = u$

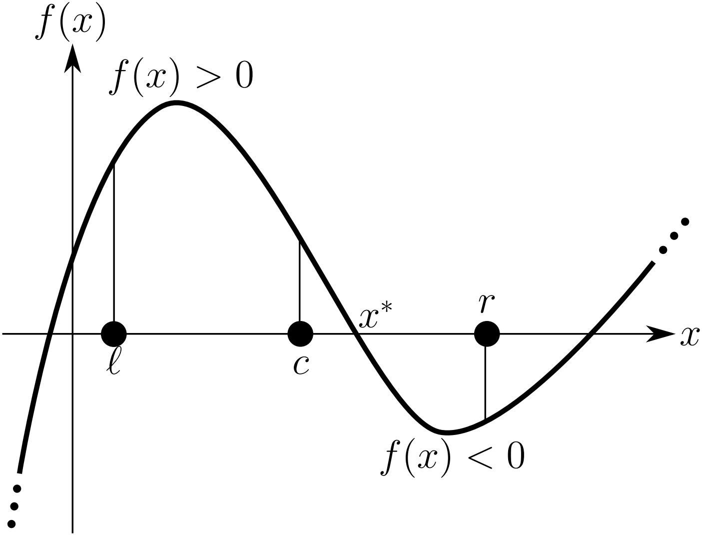
Reasonable Input

- Continuous function $f(x)$
- $\ell, r \in \mathbb{R}$ with $f(\ell) \cdot f(r) < 0$ (why?)

Bisection Algorithm

1. Compute $c = {}^{\ell+r}/_{2}$.
2. If $f(c) = 0$, return $x^* = c$.
3. If $f(\ell) \cdot f(c) < 0$, take $r \leftarrow c$. Otherwise take $\ell \leftarrow c$.
4. Return to step 1 until $|r - \ell| < \varepsilon$; then return c.

Bisection: Illustration



Two Important Questions

1. Does it converge?

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Yes! Unconditionally.

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   Yes! Unconditionally.

2. How quickly?

Convergence Analysis

Examine E_k with

$$|x_k - x^*| < E_k.$$

Bisection: Linear Convergence

$$E_{k+1} \leq \frac{1}{2} E_k$$

for $E_k \equiv |r_k - \ell_k|$

Fixed Points

$$g(x^*) = x^*$$

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Question:

Same as root-finding?

Simple Strategy

$$x_{k+1} = g(x_k)$$

Convergence Criterion

$$\begin{aligned} E_k &\equiv |x_k - x^*| \\ &= |g(x_{k-1}) - g(x^*)| \end{aligned}$$

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Convergence Criterion

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\begin{aligned}
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&= |g(x_{k-1}) - g(x^*)| \\
&\leq c|x_{k-1} - x^*| \\
&\qquad\qquad \text{if } g \text{ is Lipschitz} \\
&= cE_{k-1} \\
\implies E_k &\leq c^k E_0 \\
&\to 0 \text{ as } k \to \infty \quad (c < 1)
\end{aligned}
$$

Alternative Criterion

Lipschitz *near* x^* with good starting point.

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e.g. C^1 with $|g'(x^*)| < 1$

Convergence Rate of Fixed Point

When it converges...

Always linear (why?)

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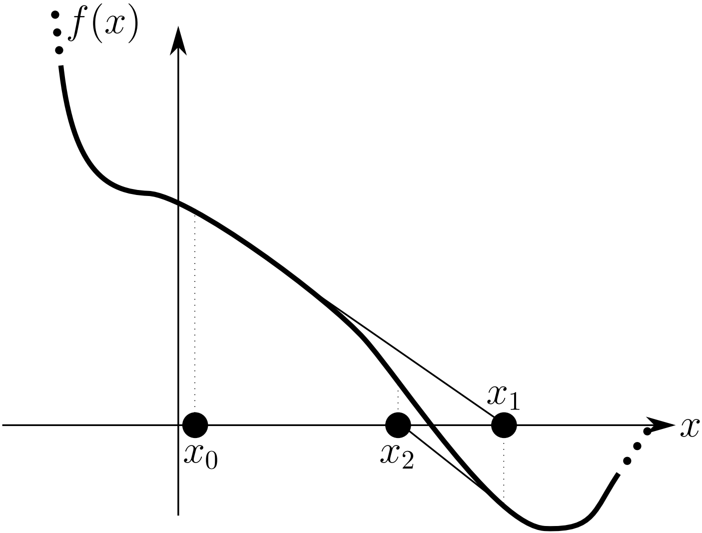
Often quadratic! ($\rightarrow$ board)

Approach for Differentiable $f(x)$



Newton's Method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

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Fixed point iteration on

$$g(x) \equiv x - \frac{f(x)}{f'(x)}$$

Convergence of Newton

Simple Root

A root x^* with $f'(x^*) \neq 0$.

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Quadratic convergence in this case! ($\rightarrow$ board)

Issue

Differentiation is hard!

Secant Method

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

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Trivia:

Converges at rate $\frac{1+\sqrt{5}}{2} \approx 1.6180339887\ldots$

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("Golden Ratio")

Hybrid Methods

Want: Convergence rate of secant/Newton with convergence guarantees of bisection

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e.g. **Dekker's Method:** Take secant step if it is in the bracket, bisection step otherwise

Single-Variable Conclusion

- Unlikely to solve exactly, so we settle for iterative methods
- Must check that method converges at all
- Convergence rates:
  - Linear: $E_{k+1} \leq CE_k$ for some $0 \leq C < 1$
  - Superlinear: $E_{k+1} \leq CE_k^r$ for some $r > 1$
  - Quadratic: $r = 2$
  - Cubic: $r = 3$
- Time *per* iteration also important

Next