

Interpolation

CS 205A: Mathematical Methods for Robotics, Vision, and Graphics

Doug James (and Justin Solomon)

Announcements

- ▶ Final homework 8 out
 - ▶ Due Thursday, March 15 (midnight)
 - ▶ No extensions – results out following day (Fri Mar 16)
- ▶ Final exam: Tuesday March 20 @ 3:30-6:30pm (Room: TBA)
- ▶ Allowed: Two double-sided pages of written notes (can reuse last one)

So Far

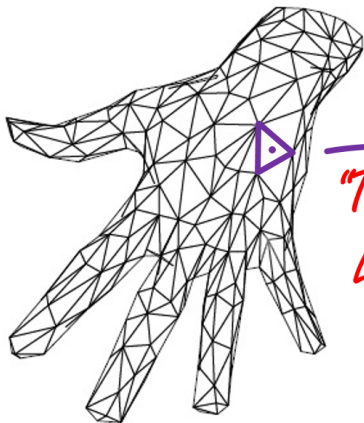
Tools for *analyzing*
functions:

Roots, minima, ...

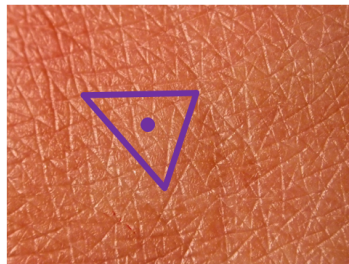
Common Situation

The function **is**
the unknown.

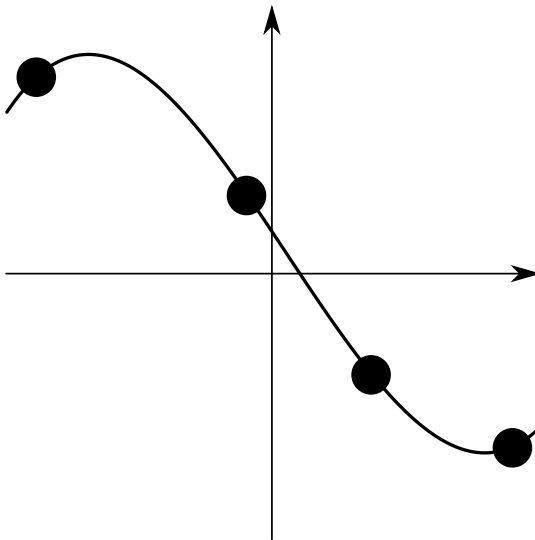
Example: Rendering



*Texture
Lookup*



Example: Regression



Input/Output

Input: $\vec{x}_i \mapsto y_i$ (exactly)

Output: $f(\vec{x})$ for $\vec{x} \notin \{\vec{x}_i\}$

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Contrast with *regression*

Initial Problem

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

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$$f : \mathbb{R} \rightarrow \mathbb{R}$$

Given: $x_i \mapsto y_i$

The Problem

$$\{f : \mathbb{R} \rightarrow \mathbb{R}\}$$

is a *huge* set.

Common Strategy

Restrict search to a basis ϕ_1, ϕ_2, \dots

$$f(x) = \sum_i a_i \phi_i(x)$$

Need to compute: \vec{a}

Monomial Basis

$$p_0(x) = 1$$

$$p_1(x) = x$$

$$p_2(x) = x^2$$

$$p_3(x) = x^3$$

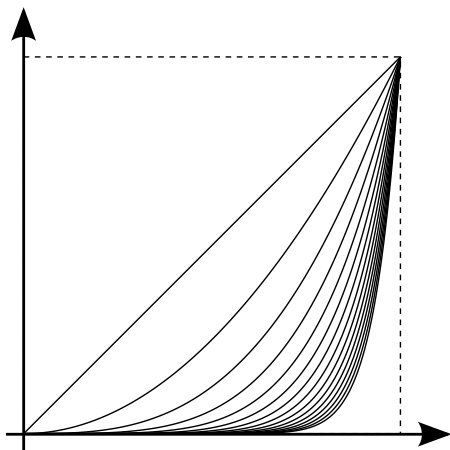
$$\vdots \quad \vdots$$

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{k-1}x^{k-1}$$

Vandermonde System

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_{k-1} & x_{k-1}^2 & \cdots & x_{k-1}^{k-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{k-1} \end{pmatrix}$$

Conditioning Issues



$p_k(x) = x^k$ look similar on $[0, 1]$ for large k

Lagrange Basis

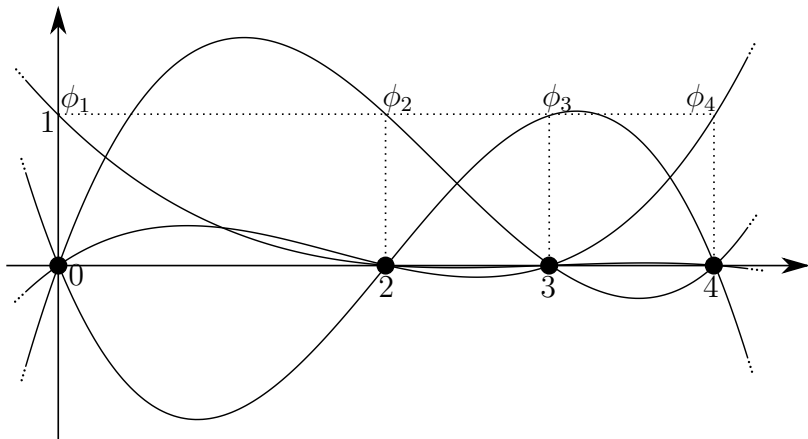
$$\phi_i(x) \equiv \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)}$$

Still polynomial!

Useful Property

$$\phi_i(x_\ell) = \begin{cases} 1 & \text{when } \ell = i \\ 0 & \text{otherwise} \end{cases}$$

Illustration of Lagrange Basis



$$x_1 = 0, x_2 = 2, x_3 = 3, x_4 = 4$$

Interpolation in Lagrange Basis

$$f(x) \equiv \sum_i y_i \phi_i(x)$$

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$O(n^2)$ time to evaluate.

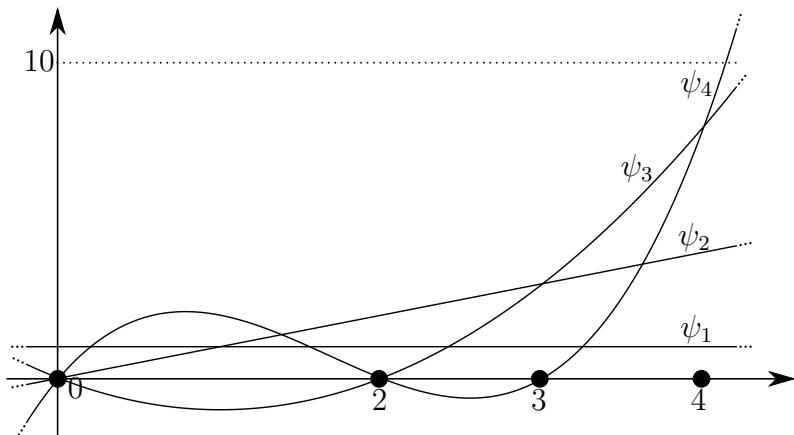
Numerical issues.

Compromise: Newton Basis

$$\psi_i(x) = \prod_{j=1}^{i-1} (x - x_j)$$

$$\psi_1(x) \equiv 1$$

Illustration of Newton Basis



$$x_1 = 0, x_2 = 2, x_3 = 3, x_4 = 4$$

Evaluating in Newton

$$f(x_1) = c_1\psi_1(x_1)$$

$$f(x_2) = c_1\psi_1(x_2) + c_2\psi_2(x_2)$$

$$f(x_3) = c_1\psi_1(x_3) + c_2\psi_2(x_3) + c_3\psi_3(x_3)$$

$$\vdots \quad \vdots$$

Triangular System

$$\begin{pmatrix} \psi_1(x_1) & 0 & 0 & \cdots & 0 \\ \psi_1(x_2) & \psi_2(x_2) & 0 & \cdots & 0 \\ \psi_1(x_3) & \psi_2(x_3) & \psi_3(x_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \psi_1(x_k) & \psi_2(x_k) & \psi_3(x_k) & \cdots & \psi_k(x_k) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{pmatrix}$$

Review:

Efficiency of solution?

Triangular System

$$\begin{pmatrix} \psi_1(x_1) & 0 & 0 & \cdots & 0 \\ \psi_1(x_2) & \psi_2(x_2) & 0 & \cdots & 0 \\ \psi_1(x_3) & \psi_2(x_3) & \psi_3(x_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \psi_1(x_k) & \psi_2(x_k) & \psi_3(x_k) & \cdots & \psi_k(x_k) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{pmatrix}$$

Review:

Efficiency of solution?

$$O(n^2)$$

Important Point

All three methods yield the
same polynomial.

Rational Interpolation

$$f(x) = \frac{p_0 + p_1x + p_2x^2 + \cdots + p_mx^m}{q_0 + q_1x + q_2x^2 + \cdots + q_nx^n}$$

Rational Interpolation

$$f(x) = \frac{p_0 + p_1x + p_2x^2 + \cdots + p_mx^m}{q_0 + q_1x + q_2x^2 + \cdots + q_nx^n}$$

$$y_i(q_0 + q_1x_i + \cdots + q_nx_i^n) = p_0 + p_1x_i + \cdots + p_mx_i^m$$

Null space problem!

Rational Interpolation

$$f(x) = \frac{p_0 + p_1x + p_2x^2 + \cdots + p_mx^m}{q_0 + q_1x + q_2x^2 + \cdots + q_nx^n}$$

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Null space problem!

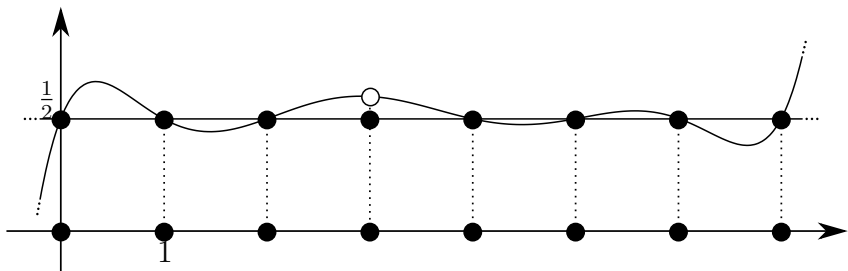
Scary example: $n = m = 1; (0, 1), (1, 2), (2, 2)$

Fourier Series

$$\cos(kx)$$

$$\sin(kx)$$

Problem with Polynomials



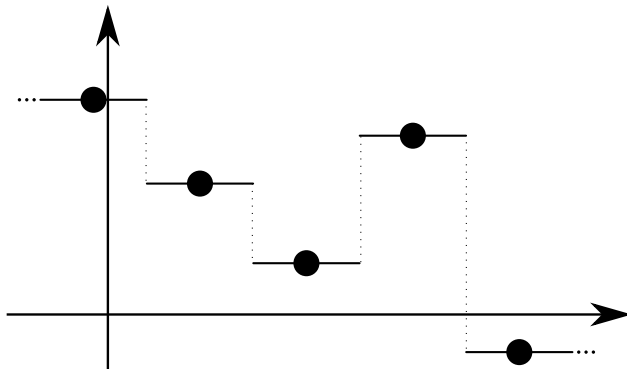
Local change can have global effect.

Compact Support

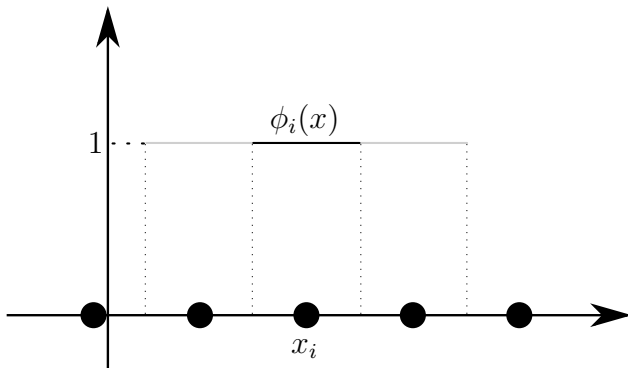
Compact support

A function $g(x)$ has *compact support* if there exists $C \in \mathbb{R}$ such that $g(x) = 0$ for any x with $|x| > C$.

Piecewise Constant Interpolation

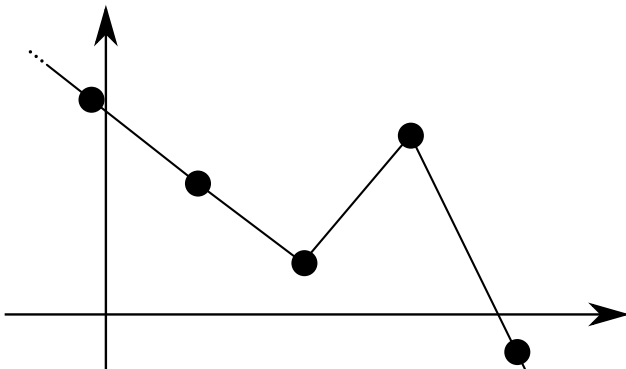


Piecewise Constant Basis

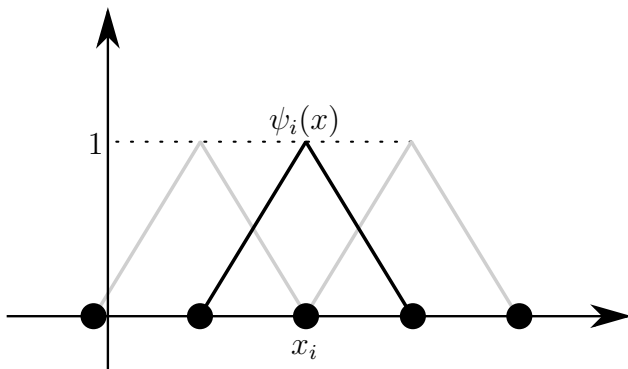


$$\phi_i(x) = \begin{cases} 1 & \text{when } \frac{x_{i-1}+x_i}{2} \leq x < \frac{x_i+x_{i+1}}{2} \\ 0 & \text{otherwise} \end{cases}$$

Piecewise Linear Interpolation



Piecewise Linear “Hat” Basis



$$\psi_i(x) = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}} & \text{when } x_{i-1} < x \leq x_i \\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & \text{when } x_i < x \leq x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Observation

Extra differentiability is possible and may look nicer but can be undesirable.

Example: Interpolating data from nonsmooth phenomena, e.g., contact.

Multidimensional Problem

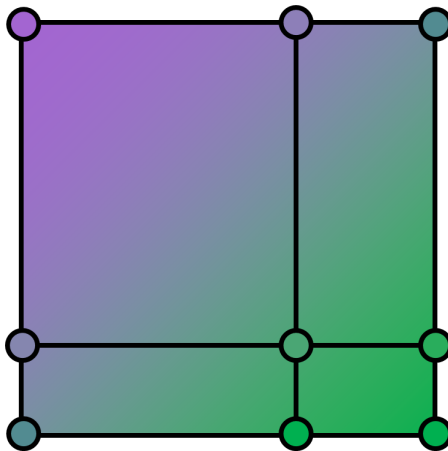
$$f : \mathbb{R}^n \rightarrow \mathbb{R}$$

Multidimensional Problem

$$f : \mathbb{R}^n \rightarrow \mathbb{R}$$

Given: $\vec{x}_i \mapsto y_i$

Interpolation on a Grid



Nearest-Neighbor Interpolation

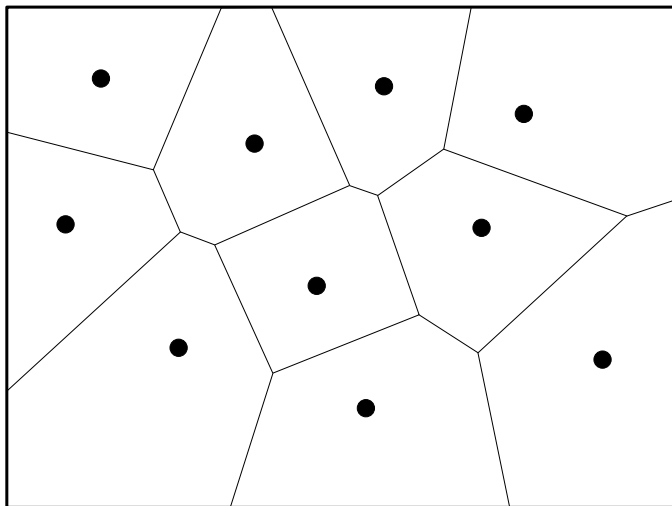
Definition (Voronoi cell)

Given $S = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\} \subseteq \mathbb{R}^n$, the *Voronoi cell* corresponding to \vec{x}_i is

$$V_i \equiv \{\vec{x} : \|\vec{x} - \vec{x}_i\|_2 < \|\vec{x} - \vec{x}_j\|_2 \text{ for all } j \neq i\}.$$

That is, it is the set of points closer to \vec{x}_i than to any other \vec{x}_j in S .

Voronoi Cells



Barycentric Interpolation

$n + 1$ points in \mathbb{R}^n

$$\sum_i a_i \vec{x}_i = \vec{x}$$

$$\sum_i a_i = 1$$

Barycentric Interpolation

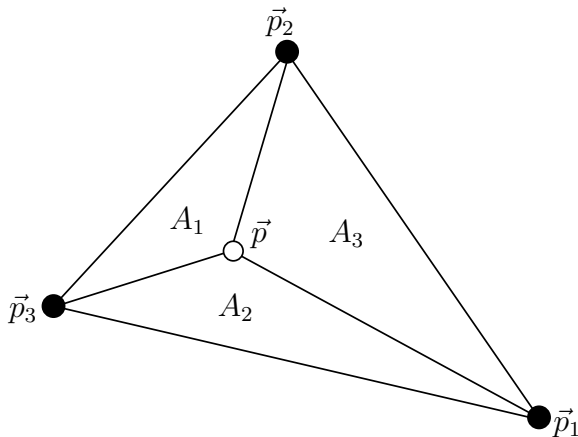
$n + 1$ points in \mathbb{R}^n

$$\sum_i a_i \vec{x}_i = \vec{x}$$

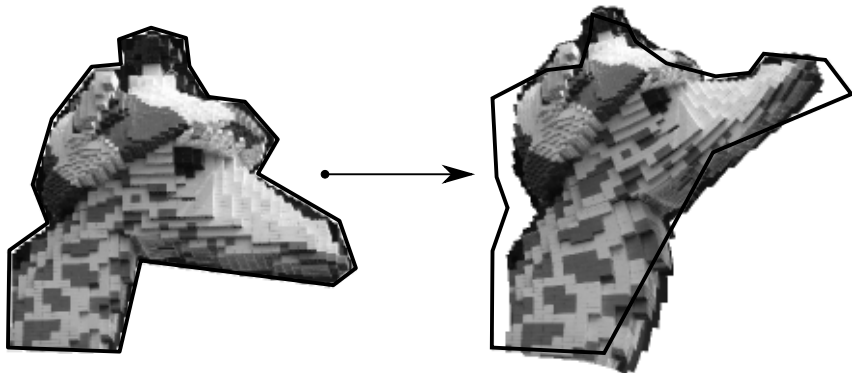
$$\sum_i a_i = 1$$

$$\longrightarrow \boxed{f(\vec{x}) = \sum_i a_i(\vec{x}) y_i}$$

Interpretation in 2D

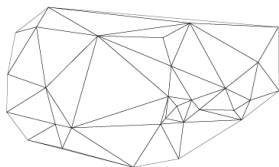


Generalized Barycentric Coordinates

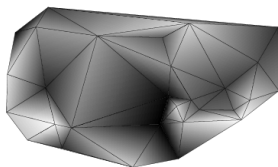


<http://www.eng.biu.ac.il/weberof/publications/complex-coordinates/>

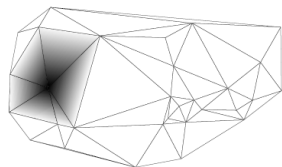
Localized Barycentric Interpolation: Triangle Hat Functions



(a) Triangle mesh



(b) Barycentric interpolation



(c) Hat function

Linear Algebra of Functions

$$\langle f, g \rangle \equiv \int_a^b f(x)g(x) dx$$

Measures “overlap” of functions!

Orthogonal Polynomials

- ▶ **Legendre:** Apply Gram-Schmidt to $1, x, x^2, x^3, \dots$
- ▶ **Chebyshev:** Same, with weighted inner product

$$w(x) = \frac{1}{\sqrt{1-x^2}}$$

Nice oscillatory properties; minimizes ringing.

Question

What is the *least-squares* approximation of f in a set of polynomials?

Piecewise Polynomial Error

▶ Piecewise constant:

$$O(\Delta x)$$

▶ Piecewise linear:

$$O(\Delta x^2)$$