Final Examination

CS 205A: Mathematical Methods for Robotics, Vision, and Graphics (Spring 2017), Stanford University

- The exam runs for 180 minutes.
- The exam contains 7 problems. **You must complete all problems.**
- The exam is closed book/notes. You may use two double-sided $8^1/_2'' \times 11''$ sheets of notes. No calculators may be used.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem and indicate that you have done so.
- Do not spend too much time on any problem. Read them all before beginning.
- Show your work, as partial credit will be awarded.

Problem	1	2	3	4	5	6	7	Total
Score (/10 each)								

The Stanford Honor Code

1. The Honor Code is an undertaking of the students, individually and collectively:

   (a) that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

   (b) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

2. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

3. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

Signature

Name

SUID

Problem 1 (True or False).

$$\text{True/False grading scheme: } \begin{cases} +1 & \text{if correct,} \\ 0 & \text{if unanswered,} \\ -1 & \text{if incorrect.} \end{cases}$$

(1) It would be better to use a *closed* Newton-Cotes integration scheme than an *open* one to approximate the integral $\int_0^1 \frac{dx}{\sqrt{x}}$.

(2) The set of points $S = \{(x, y) | x^2 + y^2 = 1\}$ is convex.

(3) A strictly convex function is also quasi-convex.

(4) The Trapezoid method is better than Heun's method for numerical integration of ODE IVPs for stiff problems.

(5) In constrained optimization, a Lagrange multiplier μ_i associated with an inequality constraint $h_i(\vec{x}) \geq 0$ is said to be active if $\mu_i \geq 0$ and inactive if $\mu_i < 0$.

(6) Golden section search may not converge for non-monotone functions.

(7) The left singular vectors $\hat{u}_i$ and right singular vectors $\hat{v}_j$ are mutually orthogonal.

(8) In the one-dimensional case, Broyden's method becomes identical to the secant method.

(9) QR factorization using Householder orthogonalization has comparable flop count to LU factorization for square matrices.

(10) For the same set of sample points, the polynomial interpolants created using Newton or Lagrange interpolating polynomials are mathematically equivalent functions.

Problem 2 (Short Answers).

(a) **Definitely integration:** Consider a composite integration scheme with interval widths Δx for which the *local* truncation error is $O(\Delta x^n)$ for some n. By what fraction should you reduce Δx to make the *global* error E become $E/10$? [3 points]

(b) **A-conjugate steepest descent:** If the initial iterate is $\vec{x}_0$, the direction of steepest descent is the residual $\vec{r}_0 = \vec{b} - A\,\vec{x}_0$. Recall that the method of steepest descent performs a line search to estimate α, such that the next iterate is $\vec{x}_1(\alpha) = \vec{x}_0 + \alpha\vec{r}_0$. At this point the previous and next residuals are orthogonal. Alternatively, derive a new α_A that corresponds to A-conjugate descent directions. [3 points]

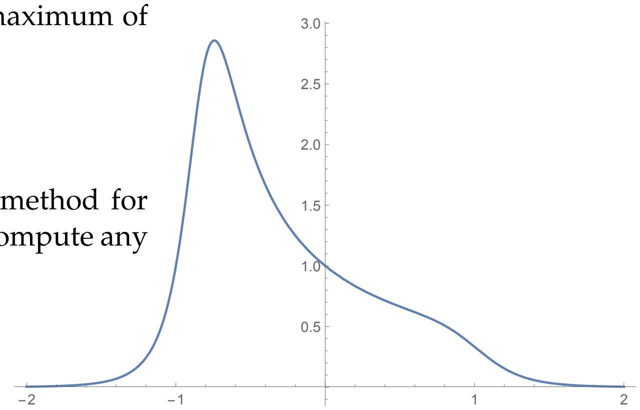
(c) **ODEs:** Convert the following explicit ODE to a first-order system of equations: [4 points]

$$y''''(t) + sin(t)y''(t) + cy'(t) + (y(t))^2 = f(t)$$

Problem 3 (Turning the Crank).

Newton's method: Consider the task of finding the maximum of the function

$$f(x) = \frac{1}{x^8 + x + 1}$$



(a) (plot shown inset). Compute one step of Newton's method for maximization using $x_0 = -1$ as your starting guess. Compute any needed derivatives by hand. [5 points]

(b) **Just one timestep**: Recall that Heun's method for the IVP ODE, $\vec{y}' = F[\vec{y}]$, is given by

$$\vec{y}_{k+1} = \vec{y}_k + \frac{h}{2}\left[F[\vec{y}_k] + F[\vec{y}_k + hF[\vec{y}_k]]\right].$$

Consider the ODE for a swinging pendulum:

$$\vec{y}'(t) = \begin{pmatrix} \theta' \\ \omega' \end{pmatrix} = F[\vec{y}] = \begin{pmatrix} \omega \\ -a \ \sin\theta \end{pmatrix}$$

where $\vec{y}(t) = (\theta(t), \omega(t))^\top$, subject to initial conditions $\vec{y}(0) = (\theta_0, \omega_0)$.
Estimate $\vec{y}(h)$ using one timestep (of size h) of Heun's method. [5 points]

Problem 4 (Matrix Computations).

(a) Show that the Householder reflection matrix $H_{\vec{v}}$ is *involutary*, meaning $H_{\vec{v}}^2 = I_{n\times n}$. [3 points]

(b) What are the eigenvalues of $H_{\vec{v}}$? [3 points]

(c) Consider a matrix-vector multiply routine that can apply some unknown-to-you $A \in \mathbb{R}^{N\times N}$ to input vectors. Given only the sequence of vectors, $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \in \mathbb{R}^N$, and their post-multiplication results, $A\vec{x}_1, A\vec{x}_2, \ldots, A\vec{x}_n$, please provide a nontrivial lower bound on the size of $\text{cond} A$. [4 points]

Problem 5 (Root Finding and Interpolation).

Inverse interpolation is a powerful technique for finding roots of $y = f(x)$ (where $x, y \in \mathbb{R}$). In the case of three function samples, $\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$, inverse quadratic interpolation cleverly fits a polynomial $p(y)$ to the inverse function, $x(y)$, then evaluates p at $y = 0$ to find the root estimate x^*.

(a) Provide a formula for $x = p(y)$ that interpolates the three samples $(y_1, x_1), (y_2, x_2), (y_3, x_3)$. [5 points]

(b) Derive a formula for the root estimate $x^* = p(0)$. [5 points]

Problem 6 (Stooge Differences).

In this course, you have learned about many good finite difference schemes for numerical differentiation. In this question, we'll explore some bad ones. Let's call them *Stooge Differences*: finite difference methods that are unnecessarily inefficient and inaccurate for our amusement.

(a) *Moe's scheme:* You know that the *forward difference formula* $\frac{f(x+h)-f(x)}{h}$ is a first-order accurate approximation to f' at the "right" point x. Moe claims that it's also a first-order accurate approximation to many "wrong" points that are $O(h)$ away. Specifically, verify that *Moe's scheme* for $f'(0)$ satisfies

$$f'_{Moe}(0) = \frac{f(2h) - f(h)}{h} = f'(0) + Ch + O(h^2)$$

for a specific C value. Using Taylor series expansions of $f(2h)$ and $f(h)$ about zero, determine the value of C. [4 points]

(b) *Curly's scheme* is also an $O(h)$ accurate scheme for $f'(0)$, but it needs *three* points! Nyuk! Nyuk! Nyuk! Unfortunately, he never wrote it down. We know it must be of the form:

$$f'_{Curly}(0) = \frac{\alpha f(h) + \beta f(2h) + \gamma f(3h)}{h}.$$

Please provide *any* nonzero values for α, β and γ that will yield an $O(h)$ scheme.
(You can state your answer here without proof–Curly didn't even write it down!) [2 points]

(c) *Larry's scheme* is an improved version of Curly's scheme that also uses three points, but, unlike Curly, he wrote it down. Also, he has a proof that the method is never better than $O(h)$ accurate. Ha! You can derive *Larry's scheme* by using the method of undetermined coefficients to find a, b and c for the following functional:

$$f'_{Larry}(0) = \frac{af(h) + bf(2h) + cf(3h)}{h}.$$

What three conditions on a, b and c are needed so that *Larry's scheme* for $f'(0)$ is precisely $O(h)$ "accurate," but not $O(h^2)$ accurate in general? [4 points]
(HINT: Expand the RHS as a Taylor series about zero.)

Problem 7 (Constrained Optimization).

(a) What are the KKT conditions of the following optimization problem? [5 points]

$$\min_{\vec{x}} \|A^\top \vec{x} - A\vec{x} - \vec{b}\|_2^2$$

subject to

$$\|\vec{x}\|_2^2 \geq \pi,$$
$$\sin(\sum_i x_i) = 0.$$

(b) **Fitting an oriented 2D bounding box:** Consider N points in 2D denoted by $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_N \in \mathbb{R}^2$. The tightest axis-aligned bounding box/rectangle is easily found by determining the max/min x and y coordinates, but may not fit as well as a rotated/oriented box (see image). Write down a constrained optimization problem for the smallest-area oriented rectangle that encloses the points. (Hint: Use the orientation angle of the box θ as a variable to help determine the best fit.) [5 points]