Eigenproblems I

CS 205A:
Mathematical Methods for Robotics, Vision, and Graphics

Doug James (and Justin Solomon)

Announcements

- Homework 1: Due tonight
- Homework 2: Out today. Control example.
- Today's class: Eigenproblem defns and examples.
- Next class: Computing eigenvalue decompositions.

Setup

Given: Collection of data points $\vec{x}_i$

- Age
- Weight
- Blood pressure
- Heart rate

Setup

Given: Collection of data points $\vec{x}_i$

- Age
- Weight
- Blood pressure
- Heart rate

Find: Correlations between different dimensions

Simplest Model

One-dimensional subspace

$$\vec{x}_i \approx c_i \vec{v}, \quad \vec{v} \text{ unknown}$$

Simplest Model

One-dimensional subspace

$$\vec{x}_i \approx c_i \vec{v}, \quad \vec{v} \text{ unknown}$$

Equivalently:

$$\vec{x}_i \approx c_i \hat{v}$$

$$\hat{v} \text{ unknown with } \|\hat{v}\|_2 = 1$$

Variational Idea

$$\text{minimize}_{\hat{v}} \sum_i \|\vec{x}_i - \text{proj}_{\hat{v}}\, \vec{x}_i\|_2^2$$

$$\text{such that } \|\hat{v}\|_2 = 1$$

Variational Idea

$$\text{minimize}_{\hat{v}} \sum_i \|\vec{x}_i - \text{proj}_{\hat{v}} \vec{x}_i\|_2^2$$
$$\text{such that } \|\hat{v}\|_2 = 1$$

What does the constraint do?

Variational Idea

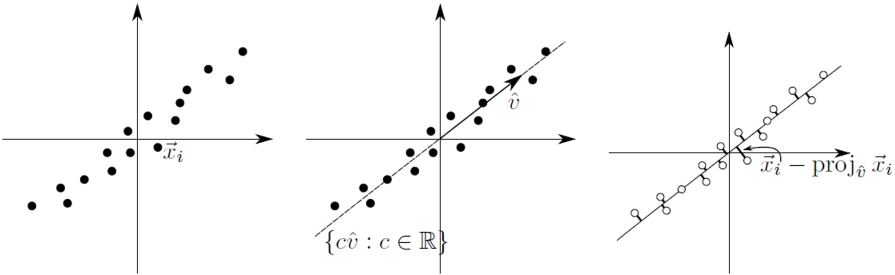
$$\text{minimize}_{\hat{v}} \sum_i \|\vec{x}_i - \text{proj}_{\hat{v}}\, \vec{x}_i\|_2^2$$

$$\text{such that } \|\hat{v}\|_2 = 1$$

What does the constraint do?

- Does not affect optimal $\hat{v}$
- Removes scaling ambiguity

Geometric Interpretation



(a) Input data (b) Principal axis (c) Projection error

Review from Last Lecture

$$\min_{c_i} \|\vec{x}_i - c_i \hat{v}\|_2$$

What is c_i?

Review from Last Lecture

$$\min_{c_i} \|\vec{x}_i - c_i\hat{v}\|_2$$

What is c_i?

$$\boxed{c_i = \vec{x}_i \cdot \hat{v}}$$

Equivalent Optimization

$$\text{maximize } \|X^\top \hat{v}\|_2^2$$
$$\text{such that } \|\hat{v}\|_2^2 = 1$$

End Goal

Eigenvector of $XX^\top$ with largest eigenvalue.

End Goal

Eigenvector of $XX^\top$ with largest eigenvalue.

"First principal component"

More after SVD!

Definitions

Eigenvalue and eigenvector

An *eigenvector* $\vec{x} \neq \vec{0}$ of $A \in \mathbb{R}^{n \times n}$ satisfies $A\vec{x} = \lambda\vec{x}$ for some $\lambda \in \mathbb{R}$; λ is an *eigenvalue*. *Complex* eigenvalues and eigenvectors instead have $\lambda \in \mathbb{C}$ and $\vec{x} \in \mathbb{C}^n$.

Definitions

Eigenvalue and eigenvector

An *eigenvector* $\vec{x} \neq \vec{0}$ of $A \in \mathbb{R}^{n\times n}$ satisfies $A\vec{x} = \lambda\vec{x}$ for some $\lambda \in \mathbb{R}$; λ is an *eigenvalue*. *Complex* eigenvalues and eigenvectors instead have $\lambda \in \mathbb{C}$ and $\vec{x} \in \mathbb{C}^n$.

Scale doesn't matter!

$\longrightarrow$ can constrain $\|\vec{x}\|_2 \equiv 1$

Eigenproblems in the Wild

- Optimize $\|A\vec{x}\|_2$ such that $\|\vec{x}\|_2 = 1$

  (important!)

Eigenproblems in the Wild

- Optimize $\|A\vec{x}\|_2$ such that $\|\vec{x}\|_2 = 1$ *(important!)*
- ODE/PDE problems: Closed solutions and approximations for $\vec{y}' = B\vec{y}$

Eigenproblems in the Wild

- Optimize $\|A\vec{x}\|_2$ such that $\|\vec{x}\|_2 = 1$ *(important!)*
- ODE/PDE problems: Closed solutions and approximations for $\vec{y}' = B\vec{y}$
- Critical points of Rayleigh quotient:

$$\frac{\vec{x}^\top A\vec{x}}{\|\vec{x}\|_2^2}$$

Two Basic Properties

Proved in textbook

Lemma

Every matrix $A \in \mathbb{R}^{n \times n}$ has at least one (complex) eigenvector.

Two Basic Properties

Proved in textbook

Lemma

Every matrix $A \in \mathbb{R}^{n \times n}$ has at least one (complex) eigenvector.

Lemma

Eigenvectors corresponding to distinct eigenvalues must be linearly independent.

Two Basic Properties

Proved in textbook

Lemma

Every matrix $A \in \mathbb{R}^{n\times n}$ has at least one (complex) eigenvector.

Lemma

Eigenvectors corresponding to distinct eigenvalues must be linearly independent.

$\longrightarrow$ at most n eigenvalues

Diagonalizability

Nondefective

$A \in \mathbb{R}^{n\times n}$ is *nondefective* or *diagonalizable* if its eigenvectors span $\mathbb{R}^n$.

Diagonalizability

Nondefective

$A \in \mathbb{R}^{n\times n}$ is *nondefective* or *diagonalizable* if its eigenvectors span $\mathbb{R}^n$.

$$D = X^{-1}AX$$

A is diagonalized by a *similarity transformation* $A \mapsto X^{-1}AX$

Definitions

Spectrum and spectral radius

The *spectrum* of A is the set of eigenvalues of A. The *spectral radius* $\rho(A)$ is the eigenvalue λ maximizing $|\lambda|$.

Extending to $\mathbb{C}^{n\times n}$

Complex conjugate

The *complex conjugate* of a number $z = a + bi \in \mathbb{C}$ is $\bar{z} \equiv a - bi$.

Extending to $\mathbb{C}^{n\times n}$

Complex conjugate

The *complex conjugate* of a number $z = a + bi \in \mathbb{C}$ is $\bar{z} \equiv a - bi$.

Conjugate transpose

The *conjugate transpose* of $A \in \mathbb{C}^{m\times n}$ is $A^H \equiv \bar{A}^\top$.

Hermitian Matrix

$$A = A^H$$

Properties

Lemma

All eigenvalues of Hermitian matrices are real.

Properties

Lemma

All eigenvalues of Hermitian matrices are real.

Lemma

Eigenvectors corresponding to distinct eigenvalues of Hermitian matrices must be orthogonal.

Spectral Theorem

Spectral Theorem

Suppose $A \in \mathbb{C}^{n\times n}$ is Hermitian (if $A \in \mathbb{R}^{n\times n}$, suppose it is symmetric). Then, A has exactly n orthonormal eigenvectors $\vec{x}_1, \cdots, \vec{x}_n$ with (possibly repeated) eigenvalues $\lambda_1, \ldots, \lambda_n$.

Spectral Theorem

Spectral Theorem

Suppose $A \in \mathbb{C}^{n\times n}$ is Hermitian (if $A \in \mathbb{R}^{n\times n}$, suppose it is symmetric). Then, A has exactly n orthonormal eigenvectors $\vec{x}_1, \cdots, \vec{x}_n$ with (possibly repeated) eigenvalues $\lambda_1, \ldots, \lambda_n$.

Full set: $D = X^\top A X$

Matrix Inverse

$$\vec{b} = c_1\vec{x}_1 + \cdots + c_k\vec{x}_k$$

$$A\vec{x} = \vec{b}$$

Matrix Inverse

$$\vec{b} = c_1\vec{x}_1 + \cdots + c_k\vec{x}_k$$

$$A\vec{x} = \vec{b}$$

$$\implies \vec{x} = \frac{c_1}{\lambda_1}\vec{x}_1 + \cdots + \frac{c_n}{\lambda_n}\vec{x}_n$$

Matrix Inverse

$$\vec{b} = c_1\vec{x}_1 + \cdots + c_k\vec{x}_k$$
$$A\vec{x} = \vec{b}$$

$$\implies \vec{x} = \frac{c_1}{\lambda_1}\vec{x}_1 + \cdots + \frac{c_n}{\lambda_n}\vec{x}_n$$

$$A = XDX^{-1} \implies A^{-1} = XD^{-1}X^{-1}$$

Matrix Square Root

- Given symmetric positive semi-definite (PSD) matrix, U
- Can compute matrix square root, $U^{1/2}$

Application: Polar decomposition

- Given real n-by-n matrix, A
- There exists a unique factorization called the Polar Decomposition

$$A = RU$$

  where R is an n-by-n orthogonal matrix, and U is an n-by-n symmetric PSD right "stretch" matrix.
- Also a left stretch matrix, W, such that $A = WR$.
- Geometric interpretation.

Application: Shape Matching

- *Fast Lattice Shape Matching* (Fast LSM)
- SIGGRAPH 2007 [Rivers and James 2007]
- `http://www.alecrivers.com/fastlsm`
- Need to compute orientation, R, of local particle groups
- Millions of polar decompositions (and eigenvalue decomps) per second

Physics (in one slide)

Newton:

$$\vec{F} = m\frac{d^2\vec{x}}{dt^2}$$

Physics (in one slide)

Newton:

$$\vec{F} = m\frac{d^2\vec{x}}{dt^2}$$

Hooke:

$$\vec{F}_s = k(\vec{x} - \vec{y})$$

First-Order System

$$M\vec{X}'' = K\vec{X}$$

$$\longrightarrow \frac{d}{dt}\begin{pmatrix} \vec{X} \\ \vec{V} \end{pmatrix} = \begin{pmatrix} 0 & I_{3n\times 3n} \\ M^{-1}K & 0 \end{pmatrix}\begin{pmatrix} \vec{X} \\ \vec{V} \end{pmatrix}$$

General ODE

$$\vec{Y}' = B\vec{Y}$$

Eigenvector Solution

$$\vec{y}' = B\vec{y}$$

$$B\vec{y}_i = \lambda_i \vec{y}_i$$

$$\vec{y}(0) = c_1\vec{y}_1 + \cdots + c_k\vec{y}_k$$

Eigenvector Solution

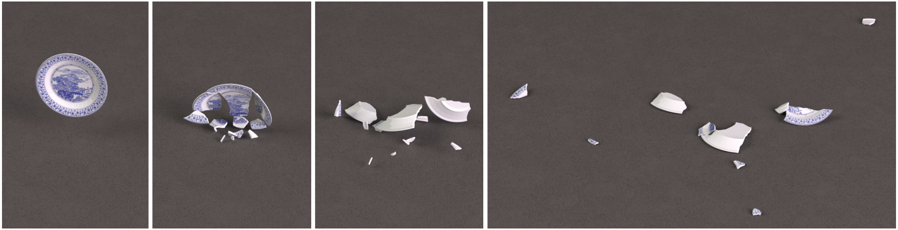
$$\vec{y}' = B\vec{y}$$

$$B\vec{y}_i = \lambda_i \vec{y}_i$$

$$\vec{y}(0) = c_1\vec{y}_1 + \cdots + c_k\vec{y}_k$$

$$\longrightarrow \vec{y}(t) = c_1 e^{\lambda_1 t}\vec{y}_1 + \cdots + c_k e^{\lambda_k t}\vec{y}_k$$

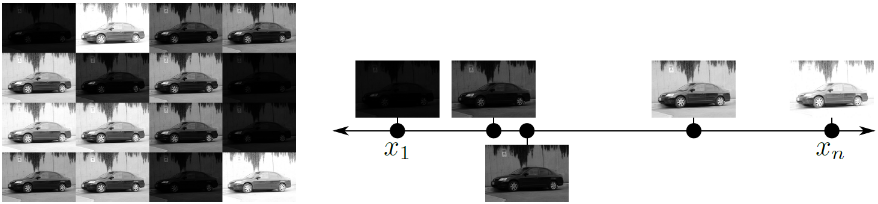
Application: Modal Sound Synthesis



Major role in physics-based sound synthesis

https://www.youtube.com/watch?v=dMUHp8i6E5E

Organizing a Collection



(a) Database of photos (b) Spectral embedding

Setup

Have: n items in a dataset

$w_{ij} \geq 0$ similarity of items i and j

$w_{ij} = w_{ji}$

Want: x_i embedding on $\mathbb{R}$

Quadratic Energy

$$E(\vec{x}) = \sum_{ij} w_{ij}(x_i - x_j)^2$$

Optimization

$$\text{minimize } E(\vec{x})$$

Optimization

$$\text{minimize } E(\vec{x})$$
$$\text{such that } \|\vec{x}\|_2^2 = 1$$

Optimization

$$\begin{aligned} \text{minimize } & E(\vec{x}) \\ \text{such that } & \|\vec{x}\|_2^2 = 1 \\ & \vec{1} \cdot \vec{x} = 0 \end{aligned}$$

Simplification

$$E(\vec{x}) = 2\vec{x}^\top(A - W)\vec{x}$$

Desired

Eigenvector of $A - W$ with **second** smallest eigenvalue.