Homework 8: Conjugate Gradients and Interpolation

CS 205A: Mathematical Methods for Robotics, Vision, and Graphics (Winter 2018)
Stanford University
Due Thursday, March 15th, before midnight (via Gradescope)

Textbook problems: 11.3 (10 points); 11.4 (10 points); 13.1 (10 points); 13.6 (10 points).

Julia Programming Assignment (60 points): In this question, we will optimize the positions of multiple robots using the Conjugate Gradient method. This question is inspired by the textbook problem 11.5.

Suppose we have a team of robots that need to communicate with each other. If two robots are nearby each other, it is cheap for them to communicate; if they are far away from each other, it is expensive for them to communicate. Each robot has its own subset of other robots it must communicate with. There is a subset of *non-mobile* robots whose positions are fixed; these robots cannot move. All other robots are *mobile* robots; these robots can move. Our goal is to solve for the position of each mobile robot, so that the total communication cost is minimized.

We model the communication requirements in this problem (i.e., which robots need to communicate with which other robots) with an undirected, unweighted graph. The nodes in the graph represent the robots, and the edges in the graph represent if two robots need to communicate. We refer to the set of all edges in the graph as E. We assume that the nodes in our graph are not connected to themselves, i.e., $(i,i) \notin E$ for each robot i. We refer to the position of the i^{th} robot as $\vec{p}_i$. We refer to the set of all non-mobile robots with the set F, and we refer to the fixed position of the k^{th} robot as $\vec{f}_k$ for each non-mobile robot $k \in F$. We model the total communication cost for our team of robots as follows,

$$\sum_{(i,j)\in E} \|\vec{p}_i - \vec{p}_j\|_2^2 \quad \text{subject to} \quad \vec{p}_k = \vec{f}_k \text{ for all } k \in F$$

(a) Assume there are N robots, and we are working in a 2D plane. Derive a system of linear equations satisfied by the optimal 2D positions of the mobile robots. The unknowns in the system should be the 2D positions of the mobile robots, so the system should have $2(N - |F|)$ unknowns. Do not represent the fixed positions of the non-mobile robots as variables in the system. Do not assume a particular graph structure in E, and do not assume a particular structure in the set of non-mobile robots F.

(b) Show that the system in (a) is symmetric and positive-definite.

(c) In a Julia notebook, construct the optimization problem above using the following parameters. Assume there are $N = 101$ robots, and they are initially positioned as follows,

$$\vec{p}_i^{\text{init}} = [t_i \cos(t_i),\ t_i \sin(t_i)]^T$$

where t_i varies linearly from 2π to 6π (inclusive). Set $F = \{20, 40, 60, 80\}$, and set $\vec{f}_k$ to be the initial position $\vec{p}_k^{\text{init}}$ for each non-mobile robot $k \in F$. Design the communication graph E as follows. Each mobile robot i must communicate with robots $\{i-5, i-4, i-3, \ldots, i+3, i+4, i+5\}$. Each non-mobile robot k must communicate with robots $\{k-10, k-9, k-8, \ldots, k+8, k+9, k+10\}$. Note that these sets should be truncated so as to avoid attempts to communicate with robots with an index less than 1 or greater than 101.

Implement both gradient descent and conjugate gradients for solving this system. Compare the number of iterations needed to reach a reasonable solution using both strategies. Plot the optimal positions you solve for, with the positions of the non-mobile robots, on a 2D scatter plot. On two separate scatter plots, show the results from gradient descent and conjugate gradients.

(d) Implement preconditioned conjugate gradients using a preconditioner of your choice. How much does convergence improve? On a third separate scatter plot, show the results from preconditioned conjugate gradients.

To simplify submission to Gradescope with your other written homework, export a single combined PDF of a clearly documented Julia notebook that shows your work, as well as your answers to the textbook questions.