Homework 3: QR and Eigenproblems

CS 205A: Mathematical Methods for Robotics, Vision, and Graphics (Winter 2018)
Stanford University
Due Thursday, February 8, 11:59pm

Textbook problems: 5.5 (20 points), 6.3 (15 points), 6.9(a) (20 points)

Guitar String Vibrations: (45 points) In this question, we will study vibrations on a string.

The Helmholtz equation is useful for modeling the standing wave vibrations on a guitar string:

$$\frac{d^2}{dx^2}y(x) + k^2 y(x) = 0. \tag{1}$$

In this equation, x is the location along the string, y is the perpendicular displacement, and $k > 0$ can be any positive real. Our string has length 1, i.e. $x \in [0,1]$. Both ends of the string are clamped, with boundary conditions $y(0) = y(1) = 0$, which should simplify our problem.

Analytical Spectral Properties of the Laplacian operator (15 points)

Let C be the set of twice-differentiable real functions $f(x)$ for $x \in (0,1)$. We can consider C a vector space: for functions $f(x)$ and $g(x)$, their sum $h(x) = f(x) + g(x)$ is in this space, and so is any scalar multiple $h(x) = af(x)$. We also define an inner product,

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx. \tag{2}$$

The Laplacian operator is therefore a linear transformation M, which takes a function $f \in C$ as input and outputs its second derivative:

$$Mf = \frac{d^2}{dx^2}f. \tag{3}$$

A1. (5 points) Verify that functions of the form $\psi_n(x) = \sqrt{2}\sin(\pi n x)$, for positive integer n, are eigenvectors of the Laplacian with eigenvalue $\lambda_n = -n^2\pi^2$. (I.e. verify $M\psi_n(x) = \lambda_n \psi_n(x)$)

A2. (5 points) Show that, as a consequence, each $\psi_n(x)$ is a solution to the Helmholtz equation above with $k = n\pi$. Verify the boundary conditions as well.

Notice that the Laplacian is symmetric: $M = M^T$, as in, for functions $f, g \in C$, $\langle f, Mg \rangle = \langle Mf, g \rangle$. (You can optionally verify this using integration by parts.) This means that the eigenvectors of M are orthogonal, and you can find them using the Q in QR iterations (problem D4).

A3. (5 points) Verify that these eigenvectors $\{\psi_n\}$ are orthonormal, i.e.

$$\int_0^1 \psi_n(x)\psi_m(x)dx = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Hint: You can use the trigonometric identity $2\sin u \sin v = \cos(u - v) - \cos(u + v)$.

Discrete Solutions (30 points)

Similar to the electric charge example in the "Designing and Analyzing Linear Systems" Julia notebook on January 18, we can discretize $y(x)$ over a grid of $N = 99$ interior values, with the first value $x_1 = 0.01$ and the last value $x_{99} = 0.99$. As before, $y(0) = y(1) = 0$. Denote the spacing as $h = \frac{1}{N+1}$. For any function $f(x)$, we will use the notation $\vec{f}$ to denote $(f(x_1), f(x_2), ..., f(x_N))^T$.

As in the notebook, we will use the following discrete 1D Laplacian matrix A of dimension $N \times N$, $(N = 99)$:

$$A_{ij} = \begin{cases} -2 & \text{if } i = j, \\ 1 & \text{if } |i - j| = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

D1. (5 points) Given that we can approximate the derivative of a function as

$$f'(x) = \frac{f(x + \frac{h}{2}) - f(x - \frac{h}{2})}{h}, \tag{6}$$

apply this equation twice to show that

$$\vec{f''} = h^{-2} A \vec{f}. \tag{7}$$

We will now use M to denote $h^{-2}A$.

D2. (5 points) Construct M and $\vec{y}$ in Julia for $y(x) = \psi_1(x)$ and verify that $\vec{y}$ satisfies the Helmholtz equation: plot $M\vec{y}$ and $-\pi^2\vec{y}$ separately, and verify that the plots are identical. (Hint: It might be more convenient, for functions you will need later, to represent M as a `SymTridiagonal`, `Tridiagonal`, or a full matrix, rather than a sparse matrix.)

D3. (5 points) Use the Julia function `eigs()` to find the eigenvectors of the Laplacian, print the 3 eigenvalues with the smallest magnitude, and plot the eigenvectors corresponding to them. The eigenvalues should be close to $-\pi^2, -4\pi^2$, and $-9\pi^2$, and the eigenvectors should look like ψ_1, ψ_2, and ψ_3, as you've shown in A1.

D4. (10 points) Since M is symmetric and its eigenvectors are orthogonal (as you've verified in A3), we can use QR iterations (book section 6.4.2) to find the eigenvectors and eigenvalues. Implement the QR iterations method from book figure 6.7, with an added step to keep track of $\overline{Q_k} = (Q_1 Q_2 ... Q_k)$, whose columns are the eigenvectors. You may use the Julia function `qr()` to get the factorization at each step. Run this for $N_{\text{it}} = 10$ iterations, print the 3 eigenvalues with the smallest magnitude, and plot the eigenvectors corresponding to them.

D5. (5 points) Plot the eigenvalues from `eigs()` from the smallest to largest, and on the same plot, also plot the eigenvalues from QR iterations for $N_{\text{it}} = 10, 30$, and 100 iterations. Briefly describe the eigenvalue plot — which eigenvalues converge first and which converge slower?