Homework 2: Linear Systems and Least Squares Problems

CS 205A: Mathematical Methods for Robotics, Vision, and Graphics (Winter 2018)
Stanford University
Due Thursday, Feb 1, before midnight (via Gradescope)

Textbook problems: 4.4 (10 points); 4.6b (10 points); 4.10 (20 points); 4.12c (10 points).

Julia Programming Assignment (50 points): In this question, we will optimize the trajectory of a 2D hovercraft.

Suppose our hovercraft is flying along a particular trajectory, but gets bumped off-course by a large unexpected gust of wind. Our goal in this question is to get back on course. There are many trajectories that will successfully get back on course, so we would like to choose a particular trajectory that is somehow optimal (we'll define exactly what optimal means below). We will need to solve for the positions, velocities, and thrust forces that will get our hovercraft back on course.

The position of our hovercraft is a 2D vector $\vec{q}$, which varies over time. The velocity of our hovercraft is $\dot{\vec{q}}$, which also varies over time. Finally, the thrust our hovercraft is exerting is a 2D vector $\vec{u}$, which varies over time. We will be solving a discretized form of this optimization problem, so we will consider the positions, velocities, and thrust forces for our hovercraft at discretely sampled moments in time. We will use the subscript notation $\vec{q}_i$ to refer to the i^{th} position sample, and likewise for velocity and thrust. The motion of our hovercraft is constrained by the following equations,

$$\vec{q}_{i+1} = \vec{q}_i + \dot{\vec{q}}_i \Delta t \qquad \dot{\vec{q}}_{i+1} = \dot{\vec{q}}_i + (\vec{u}_i + \vec{f}_g)\Delta t \tag{1}$$

where $\vec{f}_g$ is the constant force due to gravity, and Δt is the constant time interval between our discrete samples. In our problem, the initial and final states of the hovercraft are given as hard constraints, i.e.,

$$\begin{aligned} \vec{q}_1 &= \vec{q}_{\text{initial}} & \dot{\vec{q}}_1 &= \dot{\vec{q}}_{\text{initial}} & \vec{u}_1 &= \vec{u}_{\text{initial}} \\ \vec{q}_N &= \vec{q}_{\text{final}} & \dot{\vec{q}}_N &= \dot{\vec{q}}_{\text{final}} & \vec{u}_N &= \vec{u}_{\text{final}} \end{aligned} \tag{2}$$

where the right-hand sides in equation (2) are constants, and N is the number of time samples.

At each moment in time, there is some goal position $\vec{g}_i$ where we would like our hovercraft to be. We would also like our hovercraft to fly in a fuel-efficient way, so we don't want our thrust forces to be too large. We balance between our competing preferences (i.e., always being at the goal position, while also avoiding large thrust forces) by minimizing the following objective, subject to the constraints in equations (1) and (2),

$$\sum_{i=1}^{N} \left(\|\vec{q}_i - \vec{g}_i\|_2^2 + \alpha \|\vec{u}_i\|_2^2 \right) \Delta t$$

where α is a constant scalar parameter that trades off our competing preferences. In our optimization problem, the $\vec{q}_i$, $\dot{\vec{q}}_i$, and $\vec{u}_i$ variables are optimization variables; everything else is problem data.

In a Julia notebook, construct the trajectory optimization problem above using the following parameters. Use a discretization of $N = 101$ time samples, where the time t varies from $t = 0$ to $t = 10$ (inclusive). Set gravity as $\vec{f}_g = [0, -1]^T$, and set $\alpha = 1$. Set the goal trajectory as follows,

$$\vec{g}_i = \left[\left(\frac{t_i}{\pi} \right)^2 , \ \sin\left(\frac{2t_i}{\pi} \right) + \frac{t_i}{\pi} \right]^T$$

where t_i is the i^{th} time sample. Finally, set the following boundary conditions for the problem,

$$\begin{aligned} \vec{q}_1 &= \vec{g}_1 & \dot{\vec{q}}_1 &= [-3, 3]^T & \vec{u}_1 &= \vec{0} \\ \vec{q}_N &= \vec{g}_{\text{N}} & \dot{\vec{q}}_N &= \vec{0} & \vec{u}_N &= \vec{0} \end{aligned}$$

Notice that the constraint $\dot{\vec{q}}_1 = [-3, 3]^T$ models the large gust of wind that has bumped our hovercraft off course.

Express this trajectory optimization problem as a standard least squares problem with equality constraints,

$$\text{minimize} \quad \left\| A\vec{x} - \vec{b} \right\|_2^2 \quad \text{subject to} \quad C\vec{x} = \vec{d}$$

where $\vec{x}$ is a stacked vector of all our optimization variables. In your notebook, construct the matrices and vectors A, $\vec{b}$, C, $\vec{d}$ that express this trajectory optimization problem. Use the method of Lagrange multipliers to find the optimal solution to the least squares problem, and solve the resulting linear system numerically in your notebook. Show the optimal hovercraft positions and the goal positions on the same plot. On a separate plot, show the vertical and horizontal components of the optimal thrusts versus time. What is the optimal objective value?

To simplify submission to GradeScope with your other written homework, export a single combined PDF of a clearly documented Julia notebook that shows your work, as well as your answers to the textbook questions.